# MATH20132 Calculus of Several Variable. 2020-21

## **Solutions to Problems** 10 Differential forms

**1**. Given these 1-forms $\boldsymbol{\omega}$ evaluate $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t})$ at the given $\mathbf{a}$ and $t$.

i. $\boldsymbol{\omega} = (x^2 + y^2)\,dx + xydy$ at $\mathbf{a} = (1, -1)^T$ and $\mathbf{t} = (2, -1)^T$.

ii. $\boldsymbol{\omega} = 3dx + 4dy$ at

a. $\mathbf{a} = (1, -1)^T$ and $\mathbf{t} = (2, -1)^T$,
b. $\mathbf{a} = (2, 3)^T$ and $\mathbf{t} = (2, -1)^T$

**Solution** i. $\boldsymbol{\omega}_{\mathbf{a}} = 2dx(\mathbf{a}) - dy(\mathbf{a}) = 2p^1 - p^2$ so $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t}) = 2p^1(\mathbf{t}) - p^2(\mathbf{t}) = 5$.
ii. a. With $\mathbf{a} = (1, -1)^t$ we have $\boldsymbol{\omega}_{\mathbf{a}} = 3dx(\mathbf{a}) + 4dy(\mathbf{a}) = 3p^1 + 4p^2$ so $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t}) = 3p^1(\mathbf{t}) + 4p^2(\mathbf{t}) = 2$.

b. With $\mathbf{a} = (2, 3)^T$ we still have $\boldsymbol{\omega}_{\mathbf{a}} = 3dx(\mathbf{a}) + 4dy(\mathbf{a}) = 3p^1 + 4p^2$. So again $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t}) = 2$.

The point of part ii is that if the coefficients of the $dx^i$ are constant, then $\boldsymbol{\omega}_{\mathbf{a}}$ does not depend on $\mathbf{a}$.

**2.** i. Find the differential of each of the following functions as 1-forms, $\boldsymbol{\omega} : \mathbb{R}^n \to \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$, with the appropriate $n$.

a. $f(\mathbf{x}) = x \sin(x^2 y) + y$ for $\mathbf{x} \in \mathbb{R}^2$,
b. $g(\mathbf{x}) = x^4 - 3x^2y^2 + yz^2$ for $\mathbf{x} \in \mathbb{R}^3$.

ii. a. In Part i.a calculate $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t})$ with $\mathbf{a} = (2, -3)^T$ and $\mathbf{t} = (5, -2)^T$.
b. In Part i.b calculate $\boldsymbol{\omega}_{\mathbf{a}}(\mathbf{t})$ with $\mathbf{a} = (2, -3, 1)^T$ and $\mathbf{t} = (5, -2, 4)^T$.

**Solution** i.a.

$$\frac{\partial f(\mathbf{x})}{\partial x} = \sin\left(x^2 y\right) + 2x^2 y \cos\left(x^2 y\right) \quad \text{and} \quad \frac{\partial f(\mathbf{x})}{\partial y} = x^3 \cos\left(x^2 y\right) + 1.$$

Thus

$$\begin{aligned} df &= \frac{\partial f(\mathbf{x})}{\partial x}dx + \frac{\partial f(\mathbf{x})}{\partial y}dy \\ &= \left(\sin\left(x^2 y\right) + 2x^2 y \cos\left(x^2 y\right)\right) dx + \left(x^3 \cos\left(x^2 y\right) + 1\right) dy. \end{aligned}$$

b.

$$\frac{\partial g(\mathbf{x})}{\partial x} = 4x^3 - 6xy^2, \quad \frac{\partial g(\mathbf{x})}{\partial y} = -6x^2y + z^2 \quad \text{and} \quad \frac{\partial g(\mathbf{x})}{\partial z} = 2yz.$$

Thus

$$dg = \left(4x^3 - 6xy^2\right) dx + \left(-6x^2y + z^2\right) dy + 2yzdz.$$

ii. a. With $\mathbf{a} = (2, -3)^T$

$$df_{\mathbf{a}} = (\sin(-12) - 24\cos(-12))\, p^1 + (8\cos(-12) + 1)\, p^2.$$

Then with $\mathbf{t} = (5, -2)^T$

$$\begin{aligned} df_{\mathbf{a}}(\mathbf{t}) &= (\sin(-12) - 24\cos(-12))\, p^1\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\right) + (8\cos(-12) + 1)\, p^2\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix}\right) \\ &= 5(-\sin(12) - 24\cos(12)) - 2(8\cos(12) + 1) \\ &= -5\sin(12) - 136\cos(12) - 2. \end{aligned}$$

b. With $\mathbf{a} = (2, -3, 1)^T$ ,

$$dg_{\mathbf{a}} = -76p^1 + 73p^2 - 6p^3.$$

Then with $\mathbf{t} = (5, -2, 4)^T$ , we find $dg_{\mathbf{a}}(\mathbf{t}) = -550$.

**3**. In each of the following parts can you find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

i. $df = (x^2 + y^2)\, dx + 2xydy,$

ii. $df = (1 + e^x)\, dy + e^x\, (y - x)\, dx,$

iii. $df = e^y dx + x\, (e^y + 1)\, dy.$

Give your reasons and, if the function exists, write it out.

**Idea** Recall that if $f$ is Fréchet differentiable then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + ...$$

Given a form $g = g^1 dx + g^2 dy + ...$ assume there exists $f : df = g$. This means $\partial f / \partial x = g^1$.

Integrate w.r.t $x$ so $f = \int g + C$ where $C$ depends on all variables *other than* $x$.

Differentiate w.r.t. $y$ when we must have $\partial \left( \int g + C \right) / \partial y = g^2$.

Integrate w.r.t. $y$ and continue, next differentiating w.r.t the third variable. Either this process will work and you construct $f$, or you obtain a contradiction and conclude that no such $f$ exists.

**Solution** i. If $f$ exists then

$$\frac{\partial}{\partial x} f(\mathbf{x}) = x^2 + y^2$$

and so, on integrating

$$f(\mathbf{x}) = \frac{1}{3} x^3 + y^2 x + \phi(y) \,, \tag{1}$$

for some function $\phi(y)$. Differentiating this w.r.t. $y$ gives

$$\frac{\partial}{\partial y} f(\mathbf{x}) = 2yx + \phi'(y) \,.$$

This must equal the coefficient of $dy$ in the given form, i.e.

$$2yx + \phi'(y) = 2xy.$$

This can happen by choosing $\phi'(y) = 0$ so $\phi(y) = C$ for any constant $C$. Substituted back into (1) shows that

$$f(\mathbf{x}) = \frac{1}{3} x^3 + y^2 x + C$$

is a function satisfying $df = (x^2 + y^2) \, dx + 2xydy$.

ii. If $f$ exists then (be careful, $dy$ and $dx$ have been swapped!)

$$\frac{\partial}{\partial x} f(\mathbf{x}) = e^x \left( y - x \right) \quad \text{so} \quad f(\mathbf{x}) = ye^x - xe^x + e^x + \phi(y) \,, \tag{2}$$

for some function $\phi(y)$. (Integration by parts may have to have been used here.) But then

$$\frac{\partial}{\partial y} f(\mathbf{x}) = e^x + \phi'(y)$$

which equals $1 + e^x$, the coefficient of $dy$, if we choose $\phi'(y) = 1$. That is, $\phi(y) = y + C$ for any constant $C$. Substituted back into (2) and we have shown that

$$f(\mathbf{x}) = e^x\,(y - x + 1) + y + C$$

is a function satisfying $df = (1 + e^x)\,dy + e^x\,(y - x)\,dx$.

iii. If $f$ exists then

$$\frac{\partial}{\partial x} f(\mathbf{x}) = e^y \quad \text{so} \quad f(\mathbf{x}) = xe^y + \phi(y)\,,$$

for some function $\phi(y)$. But then

$$\frac{\partial}{\partial y} f(\mathbf{x}) = xe^y + \phi'(y)$$

which can only equal $x\,(e^y + 1)$, the coefficient of $dy$, if $x = \phi'(y)$, impossible, hence $f$ does **not** exist.

**4**. *In the lectures we showed that if a 1-form* $\boldsymbol{\omega}$ *is exact, i.e.* $\exists f : df = \boldsymbol{\omega}$, *then it is closed, i.e.* $\partial\omega_i/\partial x^j = \partial\omega_j/\partial x^i$ *for all pairs* $(i, j)$. *I stated that the converse is not true, i.e. not all closed forms are exact. In brief*

$$\boxed{\begin{array}{ccc} \text{exact} & \Longrightarrow & \text{closed} \\ \text{closed} & \not\Longrightarrow & \text{exact.} \end{array}}$$

In each of the following, determine whether the 1-form $\omega$ is closed, and if closed, exact. If exact, find all functions $f$ such that $df = \boldsymbol{\omega}$ :

i. $\boldsymbol{\omega} = y\,dx : \mathbb{R}^2 \to \mathrm{Hom}(\mathbb{R}^2, \mathbb{R})$;

ii. $\boldsymbol{\omega} = xy\,dx + (x^2/2)\,dy : \mathbb{R}^2 \to \mathrm{Hom}(\mathbb{R}^2, \mathbb{R})$;

iii. $\boldsymbol{\omega} = 2xydx + (x^2 + 4yz)\,dy + 2y^2dz : \mathbb{R}^3 \to \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$;

iv. $\boldsymbol{\omega} = x\,dx + xz\,dy + xy\,dz : \mathbb{R}^3 \to \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$.

**Solution** i. Since $\boldsymbol{\omega}$ is a form on $\mathbb{R}^2$ write it as $\boldsymbol{\omega} = ydx + 0dy$. Then, because

$$\frac{\partial \omega_1}{\partial x^2} = \frac{\partial y}{\partial y} = 1 \neq \frac{\partial 0}{\partial x} = \frac{\partial \omega_2}{\partial x^1},$$

the form is not closed and so it is not exact.

ii. The form is closed since

$$\frac{\partial \omega_1}{\partial x^2} = \frac{\partial\,(xy)}{\partial y} = x = \frac{1}{2}\frac{\partial x^2}{\partial x} = \frac{\partial \omega_2}{\partial x^1}.$$

If $\boldsymbol{\omega}$ is exact, so $\boldsymbol{\omega} = df$ for some differentiable $f$, then

$$\boldsymbol{\omega} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \;\text{ in which case }\; \frac{\partial f}{\partial x} = xy \;\text{ and }\; \frac{\partial f}{\partial y} = \frac{x^2}{2}. \tag{3}$$

Integrate the first of these to get

$$f = \frac{1}{2}x^2 y + g(y)\,,$$

for some function $g$. Differentiating this expression for $f$ w.r.t. $y$ gives

$$\frac{\partial f}{\partial y} = \frac{x^2}{2} + \frac{dg}{dy}; \;\text{ yet, from (3)}, \;\; \frac{\partial f}{\partial y} = \frac{x^2}{2}.$$

So we must have $dg/dy = 0$, i.e. $g = c$, a constant, in which case $f = x^2 y/2 + c$. The existence of $f$ implies that $\boldsymbol{\omega}$ is exact.

iii. The form is closed since all the following hold:

$$\begin{aligned}
\frac{\partial \omega_1}{\partial x^2} = \frac{\partial}{\partial y}2xy = 2x \;&=\; \frac{\partial}{\partial x}\left(x^2 + 4yz\right) = \frac{\partial \omega_2}{\partial x^1},\\
\frac{\partial \omega_1}{\partial x^3} = \frac{\partial}{\partial z}2xy = 0 \;&=\; \frac{\partial}{\partial x}2y^2 = \frac{\partial \omega_3}{\partial x^1},\\
\frac{\partial \omega_2}{\partial x^3} = \frac{\partial}{\partial z}\left(x^2 + 4yz\right) = 4y \;&=\; \frac{\partial}{\partial y}2y^2 = \frac{\partial \omega_3}{\partial x^2}.
\end{aligned}$$

**Note** how many conditions we need to check.

If $\boldsymbol{\omega} = df$ for some $f$ then we must have

$$\frac{\partial f}{\partial x} = 2xy, \;\; \frac{\partial f}{\partial y} = x^2 + 4yz \;\text{ and }\; \frac{\partial f}{\partial z} = 2y^2. \tag{4}$$

Integrate the first to get $f = x^2y + g(y, z)$ for some function $g$. Differentiating this w.r.t. $y$ gives

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y}; \quad \text{yet, from } (4)\,, \quad \frac{\partial f}{\partial y} = x^2 + 4yz.$$

Hence we must have

$$\frac{\partial g}{\partial y} = 4yz, \quad \text{which integrates to } g = 2y^2z + h\left(z\right),$$

for some function $h$. Combining the last two steps gives $f = x^2y+2y^2z+h\left(z\right)$. Differentiating this w.r.t. $z$ gives

$$\frac{\partial f}{\partial z} = 2y^2 + \frac{dh}{dz}; \quad \text{yet, from } (4)\,, \quad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial z} = 2y^2.$$

Therefore, we must have

$$\frac{dh}{dz} = 0, \text{ so } h = c \text{ and hence } f = x^2y + 2y^2z + c,$$

for some constant $c$. The existence of $f$ implies that $\boldsymbol{\omega}$ is exact.

iv. The form is not closed. Remember, for a form to be closed $\partial\omega_i/\partial x^j = \partial\omega_j/\partial x^i$ has to be true for **all** pairs of $(i, j)$. To **not** be closed it suffices to find **one** pair for which we do **not** have equality. In this case,

$$\frac{\partial\omega_1}{\partial x^2} = \frac{\partial x}{\partial y} = 0 \neq z = \frac{\partial\left(xz\right)}{\partial x} = \frac{\partial\omega_2}{\partial x^1}.$$

**Note** In the examples here where the form was closed it was also exact. A form that is closed but not exact has to be more complicated than those seen here. See Question 8.

**5**. Let $\boldsymbol{\omega} : U \subseteq \mathbb{R} \to \mathrm{Hom}(\mathbb{R}, \mathbb{R})$ be a 1-form on $\mathbb{R}$. This means there exists $f : U \to \mathbb{R}$ such that $\boldsymbol{\omega} = f dx$. Let $\boldsymbol{\gamma}$ be the closed interval $[a, b] \subset \mathbb{R}$ directed from $a$ to $b$. Prove that

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \int_a^b f(x)dx.$$

*This is saying that for* 1*-forms on* $\mathbb{R}$ *the integral along a line given in the lectures reduces to the previous definition of integration known from School days.*

**Hint** What parametrisation $g : [a, b] \to \boldsymbol{\gamma}$ should be chosen?

**Solution** Since $\boldsymbol{\gamma} = [a, b]$ the simplest choice of parametrisation is $g : [a, b] \to [a, b]$ the identity map, $g(x) = x$ for all $a \leq x \leq b$. Then, by definition,

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \int_a^b \boldsymbol{\omega}_{g(x)}(g'(x))\, dx.$$

Yet $\omega = f\, dx$ so, by the definition of a 1-form,

$$\boldsymbol{\omega}_{g(x)} = f(g(x))\, dx\, (g(x)) = f(x)\, p^1.$$

Also $g'(x) = 1$ for all $x$, and so

$$\boldsymbol{\omega}_{g(x)}(g'(x)) = f(x)\, p^1(1) = f(x)\,.$$

Hence

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \int_a^b \boldsymbol{\omega}_{g(x)}(g'(x))\, dx = \int_a^b f(x)\, dx,$$

as required.

(This result is normally written as $\int_{\boldsymbol{\gamma}} f dx = \int_a^b f(x)\, dx$, the left hand side being an integral of the 1-form $f dx$, the right hand side the integral of a scalar-valued function of one variable.)

**6**. Integrate the following 1-forms on the curves given.

i. $\boldsymbol{\omega} = (xz + y)\, dx + z^2 dy + xydz$ over the curve $\boldsymbol{\gamma}$ parametrised by $\mathbf{g}(t) = (t, t^2, 1+t)^T, \quad 0 \le t \le 2,$

ii. $\boldsymbol{\omega} = yzdx - xdy - (y - z)\, dz$ over the curve $\boldsymbol{\gamma}$ parametrised by $\mathbf{g}(t) = (t^2, t-1, t+1)^T$, $0 \le t \le 1$.

**Solution** i. At the point $\mathbf{g}(t)$ on the curve, the 1-form becomes the linear function

$$\boldsymbol{\omega}_{\mathbf{g}(t)} = \left(t\,(1+t) + t^2\right) p^1 + (1+t)^2\, p^2 + t^3 p^3.$$

Evaluated at the tangent vector $\mathbf{g}'(t)$ we get

$$\begin{aligned}\boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t)) &= \boldsymbol{\omega}_{\mathbf{g}(t)}\left((1, 2t, 1)^T\right) = \left(t\,(1+t) + t^2\right) + 2t\,(1+t)^2 + t^3 \\ &= 3t + 6t^2 + 3t^3.\end{aligned}$$

Hence, by definition,

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \int_0^2 \boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t))\, dt = \int_0^2 \left(3t + 6t^2 + 3t^3\right) dt = 34.$$

ii.

$$\boldsymbol{\omega}_{\mathbf{g}(t)} = (t-1)\,(t+1)\, p^1 - t^2 p^2 + 2p^3.$$

Then

$$\begin{aligned}\boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t)) &= \boldsymbol{\omega}_{\mathbf{g}(t)}\left((2t, 1, 1)^T\right) = 2t\,(t-1)\,(t+1) - t^2 + 2 \\ &= 2t^3 - t^2 - 2t + 2.\end{aligned}$$

Hence

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \int_0^1 \boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t))\, dt = \int_0^1 \left(2t^3 - t^2 - 2t + 2\right) dt = \frac{7}{6}.$$

**7**. Integrate the 1-form $\boldsymbol{\omega} = ydx + xydy$ on $\mathbb{R}^2$ around the closed curve $\boldsymbol{\gamma}$ : $x^2 + y^2 = R^2$, for a fixed $R$, in a counter-clockwise direction.

**Hint** Parametrise the curve by

$$\mathbf{g}(t) = \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix}$$

for $0 \le t \le 2\pi$. For the final integration it may save time to recall that $\int_0^{2\pi} \sin^2 t dt = \pi$.

**Solution** At the point $\mathbf{g}(t)$ on the curve, the 1-form becomes the linear function

$$\boldsymbol{\omega}_{\mathbf{g}(t)} = (R \sin t)\, p^1 + \left(R^2 \cos t \sin t\right) p^2.$$

Next,

$$\mathbf{g}'(t) = \begin{pmatrix} -R\sin t \\ R\cos t \end{pmatrix} \quad \text{so} \quad \boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t)) = -R^2 \sin^2 t + R^3 \sin t \cos^2 t.$$

Then

$$\begin{aligned} \int_{\boldsymbol{\gamma}} \boldsymbol{\omega} &= \int_0^{2\pi} \boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t))\, dt = \int_0^{2\pi} \left(-R^2 \sin^2 t + R^3 \sin t \cos^2 t\right) dt \\ &= -R^2 \int_0^{2\pi} \sin^2 t dt + \left[-R^3 \frac{\cos^3 t}{3}\right]_0^{2\pi} \\ &= -\pi R^2. \end{aligned}$$

**8**. i. Prove that the 1-form

$$\boldsymbol{\omega} = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathrm{Hom}(\mathbb{R}^2, \mathbb{R})$$

is a closed form.

ii. Let $\boldsymbol{\gamma}$ be the unit circle centre $\mathbf{0}$ in $\mathbb{R}^2$. Evaluate $\int_{\boldsymbol{\gamma}} \omega$.

iii. Deduce that $\omega$ is not exact.

This is an illustration of the result

$$\boxed{\text{closed} \not\Longrightarrow \text{exact.}}$$

**Solution** i. The form is closed because

$$\frac{\partial \omega_1}{\partial x^2} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{\partial \omega_2}{\partial x^1}.$$

ii. Parametrise $\gamma$ by

$$\mathbf{g}(t)=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix},$$

for $0\le t\le 2\pi$. Then

$$\boldsymbol{\omega}_{\mathbf{g}(t)}=-\left(\sin t\right)p^1+\left(\cos t\right)p^2,$$

and

$$\begin{aligned}\boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t)) &= -\left(\sin t\right)p^1\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}+\left(\cos t\right)p^2\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}\\ &= \sin^2 t+\cos^2 t=1.\end{aligned}$$

Hence

$$\int_{\boldsymbol{\gamma}}\boldsymbol{\omega}=\int_0^{2\pi}\boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t))\,dt=\int_0^{2\pi}dt=2\pi.$$

iii. Recall a Corollary from the notes: *If $\boldsymbol{\omega}:U\to\mathrm{Hom}\left(\mathbb{R}^n,\mathbb{R}\right)$ is an exact 1-form on an open set $U\subseteq\mathbb{R}^n$ and $\boldsymbol{\gamma}$ is a closed differentiable curve in $U$ then $\int_{\boldsymbol{\gamma}}\boldsymbol{\omega}=0$.* The contrapositive of this is if $\int_{\boldsymbol{\gamma}}\boldsymbol{\omega}\neq 0$ then $\boldsymbol{\omega}$ is not exact. In light of Part ii this can be applied to the $\boldsymbol{\omega}$ of this question, concluding that it is not exact.

(It can be shown that $\boldsymbol{\omega}$ restricted to $\mathbb{R}^2\setminus\left\{(x,0)^T: x\le 0\right\}$, i.e. the plane with the non-positive $x$-axis removed, is exact.)

**9**. i. Integrate the 1-form $\boldsymbol{\omega}=(x-z)\,dx+xyzdy+(z-y)\,dz$ along a closed path $\Gamma=\boldsymbol{\gamma}_1\cup\boldsymbol{\gamma}_2\cup\boldsymbol{\gamma}_3\cup\boldsymbol{\gamma}_4$ of four parts, each parametrised by:

- $\mathbf{g}_1(s)=(s,0,s)^T$ for $s$ from 0 to 1;
- $\mathbf{g}_2(t)=(1+t,t,1)^T$ for $t$ from 0 to 2;
- $\mathbf{g}_3(s)=(s+2,2s,s)^T$ for $s$ from 1 to 0 (note the direction of $s$);
- $\mathbf{g}_4(t)=(t,0,0)^T$ for $t$ from 2 to 0.

ii. Prove that the form $\boldsymbol{\omega}$ is not exact.

**Solution** • With $\mathbf{g}_1(s) = (s,0,s)^T$, we have $\mathbf{g}_1'(s) = (1,0,1)^T$ and

$$\boldsymbol{\omega}_{\mathbf{g}_1(s)}(\mathbf{g}_1'(s)) = sp^3\left(\mathbf{g}_1'(s)\right) = s.$$

So

$$\int_{\gamma_1} \boldsymbol{\omega} = \int_0^1 \boldsymbol{\omega}_{\mathbf{g}_1(s)}(\mathbf{g}_1'(s))\, ds = \int_0^1 sds = \frac{1}{2}.$$

• With $\mathbf{g}_2(t) = (1+t,t,1)^T$, we have $\mathbf{g}_2'(t) = (1,1,0)^T$ and

$$\begin{aligned}\boldsymbol{\omega}_{\mathbf{g}_2(t)}(\mathbf{g}_2'(t)) &= \left(tp^1 + t\,(t+1)\,p^2 + (1-t)\,p^3\right)\left(\mathbf{g}_2'(t)\right) \\ &= t + t\,(t+1) = t^2 + 2t.\end{aligned}$$

Then

$$\int_{\gamma_2} \boldsymbol{\omega} = \int_0^2 \left(t^2+2t\right) dt = \frac{20}{3}.$$

• With $\mathbf{g}_3(s) = (s+2,2s,s)^T$, we have $\mathbf{g}_3'(s) = (1,2,1)^T$ and

$$\begin{aligned}\boldsymbol{\omega}_{\mathbf{g}_3(t)}(\mathbf{g}_3'(t)) &= \left(2p^1 + 2\,(s+2)\,s^2p^2 - sp^3\right)\left(\mathbf{g}_3'(s)\right) \\ &= 2 + 4\,(s+2)\,s^2 - s \\ &= 4s^3 + 8s^2 - s + 2.\end{aligned}$$

Then, (noting in which direction $s$ is travelling),

$$\int_{\gamma_3} \boldsymbol{\omega} = \int_1^0 \left(4s^3 + 8s^2 - s + 2\right) ds = -\frac{31}{6}.$$

• With $\mathbf{g}_4(t) = (t,0,0)^T$, we have $\mathbf{g}_4'(t) = (1,0,0)^T$ and

$$\boldsymbol{\omega}_{\mathbf{g}_4(t)}(\mathbf{g}_4'(t)) = tp^1(\mathbf{g}_4'(s)) = t.$$

Then, (noting in which direction $t$ is travelling)

$$\int_{\gamma_4} \boldsymbol{\omega} = \int_2^0 tdt = -2.$$

Combining the results above

$$\int_\Gamma \boldsymbol{\omega} = \frac{1}{2} + \frac{20}{3} - \frac{31}{6} - 2 = 0.$$

ii. If $f$ satisfies $df = \boldsymbol{\omega} = (x - z)\, dx + xyzdy + (z - y)\, dz$ then $f$ must satisfy all of

$$\frac{\partial f}{\partial x} = x - z, \quad \frac{\partial f}{\partial y} = xy \quad \text{and} \quad \frac{\partial f}{\partial z} = y - z. \tag{5}$$

Integrate the first of these to get $f(\mathbf{x}) = x^2/2 - xz + g_1(y, z)$ for any function $g_1$. Differentiate w.r.t. $y$ when we get

$$\frac{\partial f}{\partial y} = \frac{\partial g_1}{\partial y}(y, z)\,.$$

Yet, when combined with (5), this gives

$$\frac{\partial g_1}{\partial y}(y, z) = xy.$$

This is impossible since there is no dependency on $x$ in the left hand side. Hence there is no $f : df = \boldsymbol{\omega}$, thus $\boldsymbol{\omega}$ is not exact.

**Note** the point of this question is to show that

$$\boxed{\begin{array}{l} \boldsymbol{\omega} \text{ exact} \implies \displaystyle\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = 0 \ \forall \text{ closed } \boldsymbol{\gamma}, \\[2ex] \exists \text{ closed } \boldsymbol{\gamma} : \displaystyle\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = 0 \not\Longrightarrow \boldsymbol{\omega} \text{ exact,} \end{array}}$$

**10**. Evaluate the 2-form

$$\left(x^2yzdx \wedge dy + (x - z)\, dx \wedge dz + yzdy \wedge dz\right)_{\mathbf{a}} (\mathbf{v}_1, \mathbf{v}_2)$$

where $\mathbf{a} = (1, -1, 2)^T$ and $\mathbf{v}_1 = (1, 2, 3)^T$, $\mathbf{v}_2 = (4, -5, 3)^T$.

**Solution**

$$\left(x^2yzdx \wedge dy + (x - z)\, dx \wedge dz + yzdy \wedge dz\right)_{\mathbf{a}} = -2p^1 \wedge p^2 - p^1 \wedge p^3 - 2p^2 \wedge p^3.$$

In turn,

$$
\begin{aligned}
p^1 \wedge p^2(\mathbf{v}_1,\, \mathbf{v}_2) &= \det \begin{pmatrix} 1 & 4 \\ 2 & -5 \end{pmatrix} = -13, \\
p^1 \wedge p^3(\mathbf{v}_1,\, \mathbf{v}_2) &= \det \begin{pmatrix} 1 & 4 \\ 3 & 3 \end{pmatrix} = -9, \\
p^2 \wedge p^3(\mathbf{v}_1,\, \mathbf{v}_2) &= \det \begin{pmatrix} 2 & -5 \\ 3 & 3 \end{pmatrix} = 21.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left(x^2yzdx \wedge dy + (x - z)\, dx \wedge dz + yzdy \wedge dz\right)_{\mathbf{a}}(\mathbf{v}_1, \mathbf{v}_2) &= -2\,(-13) - (-9) - 2\,(21) \\
&= -7.
\end{aligned}
$$

**11**. Integrate the 2-form $\boldsymbol{\beta} = yzdx \wedge dy + dx \wedge dz - (xy + 1)\, dy \wedge dz$ over the surface

$$
\mathcal{R} = \left\{ \begin{pmatrix} s+t \\ st \\ s \end{pmatrix} : 0 \leq s \leq 1, 0 \leq t \leq 2 \right\}.
$$

**Solution** Let

$$
\mathbf{g}(\mathbf{t}) = \begin{pmatrix} s+t \\ st \\ s \end{pmatrix},
$$

for $\mathbf{t} = (s,t)^T$ satisfying $0 \leq s \leq 1,\ 0 \leq t \leq 2$. Then

$$
\boldsymbol{\beta}_{\mathbf{g}(\mathbf{t})} = \left(s^2t\right) p^1 \wedge p^2 + p^1 \wedge p^3 - (st\,(s+t) + 1)\, p^2 \wedge p^3.
$$

This linear function is applied to $(d_1\mathbf{g}(\mathbf{t})\,, d_2\mathbf{g}(\mathbf{t}))$ where

$$
d_1\mathbf{g}(\mathbf{t}) = \begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix} \quad \text{and} \quad d_2\mathbf{g}(\mathbf{t}) = \begin{pmatrix} 1 \\ s \\ 0 \end{pmatrix}.
$$

Then, writing $d_1$ and $d_2$ for $d_1\mathbf{g}(\mathbf{t})\,, d_2\mathbf{g}(\mathbf{t})$ respectively,

$$\begin{aligned}
p^1 \wedge p^2(d_1, d_2) &= \det\begin{pmatrix} 1 & 1 \\ t & s \end{pmatrix} = s - t, \\
p^1 \wedge p^3(d_1, d_2) &= \det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = -1, \\
p^2 \wedge p^3(d_1, d_2) &= \det\begin{pmatrix} t & s \\ 1 & 0 \end{pmatrix} = -s.
\end{aligned}$$

Thus

$$\begin{aligned}
\boldsymbol{\beta}_{\mathbf{g}(\mathbf{t})}(d_1\mathbf{g}(\mathbf{t})\,, d_2\mathbf{g}(\mathbf{t})) &= s^2 t\,(s-t) - 1 + s\,(st\,(s+t)+1) \\
&= 2s^3 t - 1 + s.
\end{aligned}$$

Therefore the required integral is

$$\int_{\mathcal{R}} \beta = \int_0^1 \int_0^2 \left(2s^3 t - 1 + s\right) dtds = 0.$$

**12** Integrate the 2-form $\boldsymbol{\beta} = (y-1)\,dx \wedge dy$ over the region $\mathcal{D}\,(R) : x^2 + y^2 \leq R^2$ for fixed $R$.

**Hint** Parametrise the region by

$$\mathbf{g}(\mathbf{t}) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix},$$

where $\mathbf{t} = (r, \theta)$ with $0 \leq r \leq R$ and $0 \leq \theta \leq 2\pi$.

**Solution** With the parametrisation given

$$\boldsymbol{\beta}_{\mathbf{g}(\mathbf{t})} = (r\sin\theta - 1)\,p^1 \wedge p^2.$$

This linear function is applied to $(d_1\mathbf{g}(\mathbf{t})\,, d_2\mathbf{g}(\mathbf{t}))$ where

$$d_1\mathbf{g}(\mathbf{t}) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad d_2\mathbf{g}(\mathbf{t}) = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}.$$

Then

$$p^1 \wedge p^2(d_1\mathbf{g}(\mathbf{t})\,, d_2\mathbf{g}(\mathbf{t})) = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r.$$

The integral is

$$\int_{D(R)} \beta = \int_0^R r \int_0^{2\pi} (r\sin\theta - 1)\, d\theta dr = -2\pi \int_0^R r dr = -\pi R^2.$$

**13**. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ be 1 - forms given as

$\boldsymbol{\alpha} = x\, dx + yz\, dy + xyz\, dz,$

$\boldsymbol{\beta} = y^2\, dx + z\, dy - 3(x-1)\, dz$ and

$\boldsymbol{\gamma} = z\, dx \wedge dy - y\, dx \wedge dz + x\, dy \wedge dz.$

Find $\boldsymbol{\alpha} \wedge \boldsymbol{\alpha}$, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \wedge \boldsymbol{\gamma}$.

**Solution**

$$\begin{aligned}
\boldsymbol{\alpha} \wedge \boldsymbol{\alpha} &= (x\, dx + yz\, dy + xyz\, dz) \wedge (x\, dx + yz\, dy + xyz\, dz) \\
&= x^2 dx \wedge dx + xyzdx \wedge dy + x^2 yzdx \wedge dz \\
&\quad +yzxdy \wedge dx + y^2 z^2 dy \wedge dy + xy^2 z^2 dy \wedge dz \\
&\quad\quad +x^2 yzdz \wedge dx + xy^2 z^2 dz \wedge dy + x^2 y^2 z^2 dz \wedge dz \\
&= xyzdx \wedge dy + x^2 yzdx \wedge dz \\
&\quad +yzxdy \wedge dx + +xy^2 z^2 dy \wedge dz \\
&\quad\quad +x^2 yzdz \wedge dx + xy^2 z^2 dz \wedge dy
\end{aligned}$$

since $dx \wedge dx = dy \wedge dy\ dz \wedge dz = 0$.

Next $dy \wedge dx = -dx \wedge dy$, $dz \wedge dx = dx \wedge dz$ and $dz \wedge dy = -dy \wedge dz$ so

$$\begin{aligned}
\boldsymbol{\alpha} \wedge \boldsymbol{\alpha} &= xyzdx \wedge dy + x^2 yzdx \wedge dz - yzxdx \wedge dz \\
&\quad +xy^2 z^2 dy \wedge dz - x^2 yzdx \wedge dz - xy^2 z^2 dy \wedge dz \\
&= 0.
\end{aligned}$$

*In fact* $\boldsymbol{\alpha} \wedge \boldsymbol{\alpha} = 0$ *for any* 1*-forms* $\boldsymbol{\alpha}$*. Can you prove this?*

Similarly,

$$
\begin{aligned}
\boldsymbol{\alpha} \wedge \boldsymbol{\beta} &= (x\,dx + yz\,dy + xyz\,dz) \wedge \left(y^2\,dx + z\,dy - 3(x-1)\,dz\right) \\
&= xzdx \wedge dy - 3x\,(x-1)\,dx \wedge dz + y^3 zdy \wedge dx \\
&\quad -3yz\,(x-1)\,dy \wedge dz + xy^3 zdz \wedge dx + xyz^2 dz \wedge dy \\
&= xzdx \wedge dy - 3x\,(x-1)\,dx \wedge dz - y^3 zdx \wedge dy \\
&\quad -3yz\,(x-1)\,dy \wedge dz - xy^3 zdx \wedge dz - xyz^2 dy \wedge dz \\
&= \left(xz - y^3 z\right) dx \wedge dy - \left(3x\,(x-1) + xy^3 z\right) dx \wedge dz \\
&\quad - \left(3yz\,(x-1) + xyz^2\right) dy \wedge dz.
\end{aligned}
$$

Finally, since any wedge product with repeated forms is zero, e.g. $dx \wedge dy \wedge dy = 0$, we have

$$
\begin{aligned}
\boldsymbol{\alpha} \wedge \boldsymbol{\gamma} &= (x\,dx + yz\,dy + xyz\,dz) \wedge (z\,dx \wedge dy - y\,dx \wedge dz + x\,dy \wedge dz) \\
&= x^2\,dx \wedge\ dy \wedge dz - y^2 z\,dy \wedge dx \wedge dz + xyz^2\,dz \wedge dx \wedge dy \\
&= x^2\,dx \wedge\ dy \wedge dz + y^2 z\,dx \wedge dy \wedge dz + xyz^2\,dx \wedge dy \wedge dz \\
&= \left(x^2\ + y^2 z\ + xyz^2\right)\ dx \wedge dy \wedge dz
\end{aligned}
$$

We have made use of the identities $dy \wedge dx \wedge dz = -dx \wedge dy \wedge dz$ and $dz \wedge dx \wedge dy = -dx \wedge dz \wedge dy = dx \wedge dy \wedge dz$.

**14** Find the derivatives of

i. $ydx + xydy$ (seen in Question 7),

ii. $(x - z)\,dx + xyzdy + (z - y)\,dz$ (seen in Question 9),

Have you seen your answers in other questions on this sheet. If so, what conclusions can you draw?

**Hint** Think about Stokes' Theorem, surfaces and boundaries.

**Solution** i. The derivative is $(y-1)\,dx \wedge dy$, a form seen in Question 12,

ii. The derivative is $yzdx \wedge dy + dx \wedge dz - (xy+1)\,dy \wedge dz$, a form seen in Question 11.

If you look carefully at these pairs of questions 7 & 12, 9 & 11, you see the first question is a line integral over a closed curve, and the second an integral over the region within the curve. In all cases the answers are the same; they are all examples of Stokes' Theorem.

With $\boldsymbol{\gamma} : x^2 + y^2 = R^2$, for a fixed $R$, and $\mathcal{D}(R) : x^2 + y^2 \le R^2$ we have

$$\begin{aligned}
\int_{\partial\mathcal{D}} ydx + xydy &= \int_{\boldsymbol{\gamma}} ydx + xydy \qquad \text{since } \partial\mathcal{D} = \boldsymbol{\gamma} \\
&= -\pi R^2 \qquad \text{Question 7} \\
&= \int_{\mathcal{D}} (y-1)\, dx \wedge dy \qquad \text{Question 12} \\
&= \int_{\mathcal{D}} d\,(ydx + xydy)\,,
\end{aligned}$$

since $d\,(ydx + xydy) = (y-1)\, dx \wedge dy$. This final result,

$$\int_{\partial\mathcal{D}} ydx + xydy = \int_{\mathcal{D}(R)} d\,(ydx + xydy)\,,$$

is an illustration of Stoke's Theorem.

Again, the path $\Gamma$ of Question 9 is the boundary of the region $\mathcal{R}$ from Question 11, i.e. $\partial\mathcal{R} = \Gamma$. So the same argument again gives

$$\int_{\partial\mathcal{R}} (x-z)\, dx + xyzdy + (z-y)\, dz = 0 = \int_{\mathcal{R}} d\,\big(\big((x-z)\, dx + xyzdy + (z-y)\, dz\big)\big)\,.$$

Yet another illustration of Stoke's Theorem.

(Note, you may have found the integral over the arc is $-1$ times the integral over the curve. This is simply due to the order of $d_1 g$ and $d_2 g$ when evaluating the 2-form. There is a way of making this choice consistently but it depends on the 'orientation' of the surface, a subject I have not covered.)

**15** For the forms in Question 13, find $d\boldsymbol{\alpha}$, $d\boldsymbol{\beta}$ and $d\boldsymbol{\gamma}$.

**Solution**

$$\begin{aligned} d\boldsymbol{\alpha} &= dx \wedge dx + (zdy + ydz) \wedge dy + (yzdx + xzdy + xydz) \wedge dz \\ &= ydz \wedge dy + yzdx \wedge dz + xzdy \wedge dz \\ &= yzdx \wedge dz + (xz - y)\, dy \wedge dz. \end{aligned}$$

$$d\boldsymbol{\beta} = 2ydy \wedge dx + dz \wedge dy - 3dx \wedge dz = -2ydx \wedge dy - 3dx \wedge dz - dy \wedge dz.$$

$$d\boldsymbol{\gamma} = dz \wedge dx \wedge dy - dy \wedge dx \wedge dz + dx \wedge dy \wedge dz = 3dx \wedge dy \wedge dz.$$

# Solutions to Additional Questions

**16**. Integrate the 1-form $\boldsymbol{\omega} = yxdy$ along the boundary of the ellipse

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1,$$

in the counter-clockwise direction.

**Hint** to parametrise this curve use the fact that $\cos^2 t + \sin^2 t = 1$. For the final integration it may save time to note that $\int_0^{2\pi} \cos^2 t dt = \pi$.

**Solution** Set

$$\frac{x-1}{2} = \cos t \quad \text{and} \quad \frac{y+2}{3} = \sin t,$$

so the parametrisation is

$$\mathbf{g}(t) = \begin{pmatrix} 1 + 2\cos t \\ -2 + 3\sin t \end{pmatrix},$$

for $0 \le t \le 2\pi$. Then

$$\boldsymbol{\omega}_{\mathbf{g}(t)} = (-2 + 3\sin t)(1 + 2\cos t)\, p^2.$$

This linear function is applied to

$$\mathbf{g}'(t) = \begin{pmatrix} -2\sin t \\ 3\cos t \end{pmatrix},$$

when we get

$$\begin{aligned} \boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t)) &= (-2 + 3\sin t)(1 + 2\cos t)(3\cos t) \\ &= -6\cos t - 12\cos^2 t + 9\sin t\cos t + 18\sin t\cos^2 t. \end{aligned}$$

Finally, the required integral is

$$\begin{aligned} \int_0^{2\pi} \boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t))\, dt &= \int_0^{2\pi} \left(-6\cos t - 12\cos^2 t + 9\sin t\cos t + 18\sin t\cos^2 t\right) dt \\ &= \left[6\sin t + \frac{9}{2}\sin^2 t - 9\cos^3 t\right]_0^{2\pi} - 12\int_0^{2\pi} \cos^2 t dt = -12\pi, \end{aligned}$$

on using the hint.

**17**. Integrate the 1-form $\boldsymbol{\omega} = (x + y + z)\,dx + y^2 dy + xydz$ along $\boldsymbol{\gamma}$, the boundary of the unit circle in the $x$ - $y$ plane, centre the origin, in the counter-clockwise direction.

**Hint** Even though the circle lies in the $x$ - $y$ plane the 1-form is defined on $\mathbb{R}^3$ and so you have to parametrise the circle in $\mathbb{R}^3$.

**Solution** Parametrise the circle with

$$\mathbf{g}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$$

for $0 \leq t \leq 2\pi$. Then

$$\boldsymbol{\omega}_{\mathbf{g}(t)} = (\cos t + \sin t)\, p^1 + (\sin t)\, p^2 + (\cos t \sin t)\, p^3.$$

This is applied to

$$\mathbf{g}'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix},$$

when we get

$$\boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t)) = -\,(\cos t + \sin t)\sin t + \sin^2 t \cos t.$$

Thus

$$\begin{aligned}
\int_0^{2\pi} \boldsymbol{\omega}_{\mathbf{g}(t)}(\mathbf{g}'(t))\, dt &= \int_0^{2\pi} \Big(-\,(\cos t + \sin t)\sin t + \sin^2 t \cos t\Big)\, dt \\
&= \left[\frac{\cos^2 t}{2} + \frac{\sin^3 t}{3}\right]_0^{2\pi} - \int_0^{2\pi} \sin^2 t dt \\
&= -\pi.
\end{aligned}$$

Hence

$$\int_{\gamma} \boldsymbol{\omega} = -\boldsymbol{\pi}.$$

**18**. Integrate the 2-form $\boldsymbol{\beta} = -dx \wedge dy + (y-1)\, dx \wedge dz + x dy \wedge dz$ over $\mathcal{H}$, the upper half of the unit sphere, so $x^2 + y^2 + z^2 = 1$ with $z \geq 0$.

**Hint** Parametrise this surface by the spherical coordinates

$$\mathbf{g}(\mathbf{t}) = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix},$$

where $\mathbf{t} = (\phi, \theta)$, with $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$.

**Solution** With the parametrisation given

$$\boldsymbol{\beta}_{\mathbf{g}(\mathbf{t})} = -p^1 \wedge p^2 + (\sin\phi\sin\theta - 1)\, p^1 \wedge p^3 + (\sin\phi\cos\theta)\, p^2 \wedge p^3.$$

This linear function is applied to $(d_1\mathbf{g}(\mathbf{t})\,, d_2\mathbf{g}(\mathbf{t}))$ where

$$d_1\mathbf{g}(\mathbf{t}) = \begin{pmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ -\sin\phi \end{pmatrix} \quad \text{and} \quad d_2\mathbf{g}(\mathbf{t}) = \begin{pmatrix} -\sin\phi\sin\theta \\ \sin\phi\cos\theta \\ 0 \end{pmatrix}.$$

Then, writing $d_1$ for $d_1\mathbf{g}(\mathbf{t})$ and $d_2$ for $d_2\mathbf{g}(\mathbf{t})$,

$$\begin{aligned}
p^1 \wedge p^2(d_1, d_2) &= \det\begin{pmatrix} \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \cos\phi\sin\theta & \sin\phi\cos\theta \end{pmatrix} = \cos\phi\sin\phi \\
p^1 \wedge p^3(d_1, d_2) &= \det\begin{pmatrix} \cos\phi\cos\theta & -\sin\phi\sin\theta \\ -\sin\phi & 0 \end{pmatrix} = -\sin^2\phi\sin\theta, \\
p^2 \wedge p^3(d_1, d_2) &= \det\begin{pmatrix} \cos\phi\sin\theta & \sin\phi\cos\theta \\ -\sin\phi & 0 \end{pmatrix} = \sin^2\phi\cos\theta.
\end{aligned}$$

Thus

$$\begin{aligned}
\boldsymbol{\beta}_{\mathbf{g}(\mathbf{t})}(d_1, d_2) &= -\cos\phi\sin\phi - (\sin\phi\sin\theta - 1)\sin^2\phi\sin\theta + \sin\phi\cos\theta\sin^2\phi\cos\theta \\
&= -\cos\phi\sin\phi - \sin^3\phi\sin^2\theta + \sin^2\phi\sin\theta + \sin^3\phi\cos^2\theta.
\end{aligned}$$

Then the integral is

$$\int_0^{\pi/2}\int_0^{2\pi} \left(-\cos\phi\sin\phi - \sin^3\phi\sin^2\theta + \sin^2\phi\sin\theta + \sin^3\phi\cos^2\theta\right) d\theta d\phi. \tag{6}$$

The inner integral equals

$$
\begin{aligned}
&-2\pi\cos\phi\sin\phi-\sin^3\phi\int_0^{2\pi}\sin^2\theta d\theta+\sin^2\phi\int_0^{2\pi}\sin\theta d\theta+\sin^3\phi\int_0^{2\pi}\cos^2\theta d\theta\\
&=-2\pi\cos\phi\sin\phi-\pi\sin^3\phi+\pi\sin^3\phi\\
&=-2\pi\cos\phi\sin\phi,
\end{aligned}
$$

since

$$\int_0^{2\pi}\sin^2\theta d\theta=\int_0^{2\pi}\cos^2\theta d\theta=\pi \quad\text{and}\quad \int_0^{2\pi}\sin\theta d\theta=0.$$

Then the double integral (6) is

$$-2\pi\int_0^{\pi/2}\cos\phi\sin\phi d\phi=-\pi\left[\sin^2\phi\right]_0^{\pi/2}=-\pi.$$

Thus

$$\int_{\mathcal{H}}\beta=-\pi.$$

**19**. Integrate the 2-form $\boldsymbol{\omega}=\left(x^2y+y^2z^2\right)dx\wedge dy+y^3zdx\wedge dz+xy^2zdy\wedge dz$ over the surface of the sphere $x^2+y^2+z^2=a$.

**Solution** We use the parametrization of the previous question, i.e.

$$\begin{pmatrix} a\sin\phi\cos\theta\\ a\sin\phi\sin\theta\\ a\cos\phi\end{pmatrix},$$

this time with $-\pi/2\le\phi\le\pi/2$ and $0\le\theta\le 2\pi$.

Use the results on $p^i\wedge p^l$ from the last question we find that the integral over $\left(x^2y+y^2z^2\right)dx\wedge dy$ is one of

$$
\begin{aligned}
&\left((a\sin\phi\cos\theta)^2(a\sin\phi\sin\theta)+(a\sin\phi\sin\theta)^2\left(a\sin\phi\sin\theta\right)^2\right)\left(a^2\cos\phi\sin\phi\right)\\
&\qquad=a^5\left(\sin^3\phi\cos\phi\sin\theta\cos^3\theta+\sin^5\phi\cos\phi\sin^4\theta\right)
\end{aligned}
$$

In both terms the integrals over $\phi$ will give zero.

The term $y^3zdx\wedge dz$ becomes

$$(a\sin\phi\sin\theta)^3a\cos\phi\left(-a^2\sin^2\phi\sin\theta\right)=a^6\sin^5\phi\cos\phi\sin^4\theta.$$

Again the integral over $\phi$ will give zero.

Finally, the $xy^2zdy \wedge dz$ term gives

$$a \sin\phi \cos\theta \left(a \sin\phi \sin\theta\right)^2 a \cos\phi \left(a^2 \sin^2\phi \cos\theta\right)^2 = a^8 \sin^7\phi \cos\phi \sin^2\theta \cos^3\theta.$$

Yet again the integral over $\phi$ gives zero. Hence the complete integral over the surface of the sphere is zero.

A quicker proof of this follows from the Divergence Theorem, see Appendix of Notes for it's statement. Let $S \subseteq \mathbb{R}^3$ be a three dimensional subset with a boundary $\partial S$, a two dimensional surface. Let $\boldsymbol{\omega} = f^1 dy \wedge dz + f^2 dz \wedge dx + f^3 dx \wedge dy$ be a two form (note the ordering of the terms). Then the Divergence Theorem asserts that

$$\int_{\partial S} \boldsymbol{\omega} = \int_S (\mathrm{div}\mathbf{f})\, dxdydz$$

where

$$\mathrm{div}\mathbf{f} = \frac{\partial f^1}{\partial x} + \frac{\partial f^2}{\partial y} + \frac{\partial f^3}{\partial z}.$$

In the present example

$$\boldsymbol{\omega} = xy^2zdy \wedge dz - y^3zdz \wedge dx + \left(x^2y + y^2z^2\right) dx \wedge dy,$$

and it is easily seen that $\mathrm{div}\mathbf{f} = 0$.

**20**. Integrate the 2-form $\boldsymbol{\beta} = -dx \wedge dy + (y-1)\, dx \wedge dz + xdy \wedge dz$ over $\mathcal{D}$, the region $x^2 + y^2 \le 1$ in the $x$ - $y$ plane.

**Hint** As in question 17, though the region of integration lies in the $x$ - $y$ plane the form is defined on $\mathbb{R}^3$ and so you have to choose a parametrisation of the region as a subset of $\mathbb{R}^3$.

**Solution** Parametrise the region by

$$\mathbf{g}(\mathbf{t}) = \begin{pmatrix} r\cos t \\ r \sin t \\ 0 \end{pmatrix},$$

where $\mathbf{t} = (r,t)^T$ satisfies $0 \le r \le 1, 0 \le t \le 2\pi$. Then

$$\boldsymbol{\beta}_{\mathbf{g}(\mathbf{t})} = -p^1 \wedge p^2 + (r \sin t - 1)\, p^1 \wedge p^3 + (r\cos t)\, p^2 \wedge p^3.$$

This linear function is applied to $(d_1\mathbf{g}(\mathbf{t}), d_2\mathbf{g}(\mathbf{t}))$ where

$$d_1\mathbf{g}(\mathbf{t}) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} \quad \text{and} \quad d_2\mathbf{g}(\mathbf{t}) = \begin{pmatrix} -r\sin t \\ r\cos t \\ 0 \end{pmatrix}.$$

Because both vectors have 0 in their last position, only $p^1 \wedge p^2(d_1\mathbf{g}(\mathbf{t}), d_2\mathbf{g}(\mathbf{t}))$ is non-zero. In fact

$$\boldsymbol{\beta}_{\mathbf{g}(\mathbf{t})}(d_1\mathbf{g}(\mathbf{t}), d_2\mathbf{g}(\mathbf{t})) = -r.$$

Hence the required integral is

$$-\int_0^1 \int_0^{2\pi} r dt dr = -\pi.$$

Hence

$$\int_{\mathcal{D}} \beta = -\pi.$$

**21**. Explain why Questions 17, 18 and 20 together illustrate Stoke's Theorem.

**Solution** The derivative of the 1-form from Question 17 is the 2-form found in both Questions 18 and 20. Also, the boundaries of the regions $\mathcal{H}$ and $\mathcal{D}$ in Questions 18 and 20 is the path in Question 17.

For example, Questions 17 & 18 give

$$\begin{aligned}
\int_{\partial\mathcal{H}} \boldsymbol{\omega} &= \int_{\gamma} \boldsymbol{\omega} && \text{since } \partial\mathcal{H} = \gamma \\
&= -\pi && \text{Question 17} \\
&= \int_{\mathcal{H}} \boldsymbol{\beta} && \text{Question 18} \\
&= \int_{\mathcal{H}} d\boldsymbol{\omega} && \text{since } d\boldsymbol{\omega} = \boldsymbol{\beta}.
\end{aligned}$$

The final

$$\int_{\partial\mathcal{H}} \boldsymbol{\omega} = \int_{\mathcal{H}} d\boldsymbol{\omega}$$

is simply Stoke's Theorem.

**22**. Integrate the form $\boldsymbol{\beta} = ydx \wedge dy$ over the area within the ellipse

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1.$$

**Hint** parametrise this region by

$$\mathbf{g}(\mathbf{t}) = \begin{pmatrix} 1 + 2r\cos t \\ -2 + 3r\sin t \end{pmatrix}$$

where $\mathbf{t} = (r,t)^T$ satisfies $0 \le r \le 1, 0 \le t \le 2\pi$.

**Note** this question is related to Question 16 by Stoke's Theorem.

**Solution** With the parametrisation given,

$$\boldsymbol{\omega}_{\mathbf{g}(t)} = (-2 + 3r\sin t)\, p^1 \wedge p^2.$$

This linear function is applied to $d_1\mathbf{g}(t)$ and $d_2\mathbf{g}(t)$ which are

$$d_1\mathbf{g}(t) = \begin{pmatrix} 2\cos t \\ 3\sin t \end{pmatrix} \quad \text{and} \quad d_2\mathbf{g}(t) = \begin{pmatrix} -2r\sin t \\ 3r\cos t \end{pmatrix}.$$

Then

$$p^1 \wedge p^2(d_1\mathbf{g}(t)\,, d_2\mathbf{g}(t)) = \det\begin{pmatrix} 2\cos t & -2r\sin t \\ 3\sin t & 3r\cos t \end{pmatrix} = 6r.$$

Thus the required integral is

$$6\int_0^1 r\left(\int_0^{2\pi} (-2 + 3r\sin t)\, dt\right) dr = -24\pi\int_0^1 rdr = -12\pi.$$

*If you have been reading the asides in my notes on Vector Calculus the following may be of interest.*

**23**. Suppose that $\mathbf{f}$, $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^3$ are two vector fields on $\mathbb{R}^3$. Recall, from the asides in the notes, the vectors

$$d\mathbf{r} = \begin{pmatrix} dx^1 \\ dx^2 \\ \vdots \\ dx^n \end{pmatrix} \quad \text{and} \quad \mathbf{n} = \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix}.$$

Prove that

$$(\mathbf{f} \bullet d\mathbf{r}) \wedge (\mathbf{g} \bullet d\mathbf{r}) = \mathbf{f} \times \mathbf{g} \bullet \mathbf{n}.$$

We say that $\mathbf{f} \bullet d\mathbf{r}$ and $\mathbf{g} \bullet d\mathbf{r}$ are the 1-forms associated with $\mathbf{f}$ and $\mathbf{g}$ while $\mathbf{f} \times \mathbf{g} \bullet \mathbf{n}$ is the 2-form associated with $\mathbf{f} \times \mathbf{g}$. Hence this result says that the wedge product of the 1-forms associated with $\mathbf{f}$ and $\mathbf{g}$ is the 2-form associated with the vector product of $\mathbf{f}$ and $\mathbf{g}$.

**Solution** Let $\mathbf{f} = (f^1, f^2, f^3)^T$ and $\mathbf{g} = (g^1, g^2, g^3)^T$. Then

$$\mathbf{f} \times \mathbf{g} = \left(f^2 g^3 - f^3 g^2,\ f^3 g^1 - f^1 g^3,\ f^1 g^2 - f^2 g^1\right)^T,$$

and

$$\begin{aligned} \mathbf{f} \bullet d\mathbf{r} &= f^1 dx^1 + f^2 dx^2 + f^3 dx^3 \\ \mathbf{g} \bullet d\mathbf{r} &= g^1 dx^1 + g^2 dx^2 + g^3 dx^3. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{f} \times \mathbf{g} \bullet \mathbf{n} &= \left(f^2 g^3 - f^3 g^2\right) dy \wedge dz + \left(f^3 g^1 - f^1 g^3\right) dz \wedge dx \\ &\quad + \left(f^1 g^2 - f^2 g^1\right) dx \wedge dy. \end{aligned} \tag{7}$$

Also,

$$\begin{aligned} (\mathbf{f} \bullet d\mathbf{r}) \wedge (\mathbf{g} \bullet d\mathbf{r}) &= \left(f^1 dx^1 + f^2 dx^2 + f^3 dx^3\right) \wedge \left(g^1 dx^1 + g^2 dx^2 + g^3 dx^3\right) \\ &= f^1 dx^1 \wedge g^1 dx^1 + f^1 dx^1 \wedge g^2 dx^2 + f^1 dx^1 \wedge g^3 dx^3 \\ &\quad + f^2 dx^2 \wedge g^1 dx^1 + f^2 dx^2 \wedge g^2 dx^2 + f^2 dx^2 \wedge g^3 dx^3 \\ &\quad + f^3 dx^3 \wedge g^1 dx^1 + f^3 dx^3 \wedge g^2 dx^2 + f^3 dx^3 \wedge g^3 dx^3 \\ &= f^1 g^2 dx^1 \wedge dx^2 + f^1 g^3 dx^1 \wedge dx^3 + f^2 g^1 dx^3 \wedge dx^1 \\ &\quad + f^2 g^3 dx^2 \wedge dx^3 + f^3 g^1 dx^3 \wedge dx^1 + f^3 g^2 dx^3 \wedge dx^2 \\ &= \left(f^1 g^2 - f^2 g^1\right) dx^1 \wedge dx^2 - \left(f^3 g^1 - f^1 g^3\right) dx^1 \wedge dx^3 \\ &\quad + \left(f^2 g^3 - f^3 g^2\right) dx^2 \wedge dx^3 \\ &= \mathbf{f} \times \mathbf{g} \bullet \mathbf{n} \end{aligned}$$

as required. The last step requires a rearrangement of (7).